Problem 11)

$$\text{Jacobian} = \left\| \begin{matrix} \partial x/\partial r & \partial y/\partial r & \partial z/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta & \partial z/\partial\theta \\ \partial x/\partial\phi & \partial x/\partial\phi & \partial x/\partial\phi \end{matrix} \right\| = \left\| \begin{matrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{matrix} \right\|$$

$$= \sin\theta\cos\phi\,(r^2\sin^2\theta\cos\phi) - \sin\theta\sin\phi\,(-r^2\sin^2\theta\sin\phi)$$

$$+ \cos\theta\,(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi)$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\,\underbrace{(\cos^2\phi + \sin^2\phi)}_{1}$$

$$= r^2\sin^3\theta\,\underbrace{(\cos^2\phi + \sin^2\phi)}_{1} + r^2\sin\theta\cos^2\theta$$

$$= r^2\sin\theta\,\underbrace{(\sin^2\theta + \cos^2\theta)}_{1} = r^2\sin\theta.$$

A volume integral such as $\iiint f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ may thus be written equivalently as follows:

$$\iiint \tilde{f}(r,\theta,\phi)\overbrace{r^2\sin\theta}^{\text{Jacobian}}\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

Note that we have used $\tilde{f}(r,\theta,\phi)$ for the function $f(x,y,z)$ when the arguments x,y,z are replaced with $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$.